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Representations of low-rank orthosymplectic superalgebras by superfield techniques

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Abstract. Representations of Lie superalgebras may be realised as functions (superfields) on graded manifolds (suitably chosen coset spaces). The technique is illustrated for OSp(1/2) and OSp(2/2) with a review of well known results, and applied in the cases of OSp(3/2) and OSp(4/2) to construct some classes of irreducible representations.

1. Introduction and main results

Following the realisation of the significance of supersymmetry in mathematics and physics (Corwin *et al* 1975) and the classification of all simple superalgebras (see, for example, Kac 1977, Rittenberg 1978, Scheunert 1979 and references therein), a programme of investigating the representations of the (classical and exceptional) superalgebras has developed along several lines. As well as the general theory (Kac 1978), there have been a number of case studies (for example, Corwin *et al* 1975, Scheunert *et al* 1977, Bednář and Šachl 1978, 1979, Marcu 1980a, b). Other approaches have been via graded Young diagrams (Dondi and Jarvis 1981, see also Jarvis and Green 1979, Green and Jarvis 1982), tensor products (Ne'eman and Sternberg 1980), tensor and supercharacter methods (Balantekin and Bars 1981a, b, 1982) and using Gel'fand patterns (Sun and Han 1980, Han *et al* 1980, Han 1981). In recent work Hurni and Morel (1981, 1982, see also Morel and Thierry-Mieg 1981) have applied general weight-space techniques (Kac 1978) to representations of superalgebras in a general framework. Infinite-dimensional representations have been studied by Edwards (1980), Blank *et al* (1981) and Hughes (1981).

Major current physical applications are in supersymmetry and supergravity (see, for example, Fayet and Ferrara 1977, van Nieuwenhuizen 1981); the superalgebras of concern here are, of course, not simple in general. These do arise in nuclear models of, for example, U(6/4) (Iachello 1980, Balantekin *et al* 1981); similar supergroups have appeared in solutions to anomaly constraints (Banks *et al* 1980). In the realm of internal supersymmetry following work on SU(2/1) for the electroweak model (Ne'eman 1979), and SU(1/1) for free electromagnetism (Jarvis 1982), the supergroup SU(5/1) has been used as an internal classification group (Taylor 1979, Dondi and Jarvis 1980) extended to SU($5+k$/1) for several generations (Ne'eman and Sternberg 1980). Finally a space–time, but non-fermionic, supersymmetry seems to underly the BRS invariance of quantum gauge fields and their compensating ghost counterparts:

in one formulation (Delbourgo and Jarvis 1982) a type of dimensional reduction has been used with a space–time supergroup OSp(4/2).

It is with such applications in mind that the present study of representations of low-rank superalgebras, specifically OSp(3/2) and OSp(4/2) (the cases OSp(1/2) and OSp(2/2) are well known) is directed. The aim (partially achieved in this paper—see the summary below) has been to provide a comprehensive list of finite-dimensional irreducible representations (both typical and atypical) of these superalgebras. This would complement other methods and extend the tensor and Young-diagram approaches by serving as a guide to the spinor representations and those representations which cannot be decomposed. Finally, the cases discussed provide simple examples of graded manifolds: thus in the OSp(2/2) example (see § 2.2) the space chosen may be regarded as a 1/2 graded version of the three-sphere.

Our method is that of induced representations. For a supergroup G and subgroup H, with corresponding superalgebras $\mathscr{G}$ and $\mathscr{H}$, representations of $\mathscr{G}$ are afforded by functions Φ on G/H taking their values in a representation space $\mathscr{V}$ of $\mathscr{H}$. Indeed if $x, y, z, \ldots$ are a set of coset representatives of G/H then for $g \in G$ the group action in an appropriate basis for $\mathscr{V}$ is

$$(g\Phi)_a(x) = \hat{h}_a^b \Phi_b(y) \tag{1}$$

where y is the unique coset representative such that $g \cdot x = yh^{-1}$, $h \in H$ and $\hat{h}_a^b$ is the matrix representing h in the chosen basis for $\mathscr{V}$.

Specifically, we choose coset representatives of the form $\exp \Sigma\,(xX + \theta Q)$, where X and Q are generic even and odd elements of $\mathscr{G}/\mathscr{H}$. To allow the exponential map to be defined (at least formally), θ must be an a-number (anticommuting) parameter and x a c-number as usual. If now S is an odd element of $\mathscr{G}$ and η an a-number parameter then the group action on G/H is infinitesimally

$$\exp(\eta S) \exp \sum (xX + \theta Q) = \exp \sum [(x + \eta\theta f(x, \theta^2))X + (\theta + \eta g(x, \theta^2))Q] \exp \sum (\eta k(x, \theta)K) \tag{2}$$

where $K \in \mathscr{H}$. The particular basis chosen will dictate the precise form of $f(x, \theta^2)$, $g(x, \theta^2)$; for appropriate $\mathscr{H}$ they may be restricted to polynomials of low degree, and may be obtained directly via the BCH formula. From (1) and (2) the corresponding differential representation will be

$$S \to \sum [f(x, \theta^2)\theta\, \partial/\partial x + g(x, \theta^2)\, \partial/\partial\theta - k(x, \theta)\hat{K}] \tag{3}$$

where now $\hat{K}$ is the matrix of the infinitesimal generator K in the representation carried by $\mathscr{V}$. Often it will be possible to decompose $\mathscr{H}$ as $= \mathscr{H}_0 + \mathscr{H}_+$, where $\mathscr{H}_+$ is an ideal (invariant subalgebra, $[\mathscr{H}, \mathscr{H}_+] \subset \mathscr{H}_+$). Representations of $\mathscr{H}_0$ are then easily extended to $\mathscr{H}$ by taking them to be zero on $\mathscr{H}_+$.

The action on superfields corresponding to (2) is given simply by

$$\delta\Phi(x, \theta) = S\Phi(x, \theta).$$

The representation (with matrix elements obtainable by expanding in x and (polynomially) in θ) is in general infinite-dimensional, but with a finite-dimensional factor related to the choice of $\mathscr{V}$. From Kac (1978), all irreducible representations can be obtained by choosing $\mathscr{H}$ as a Borel subalgebra and $\mathscr{H}_0$ a Cartan subalgebra (and $\mathscr{V}$ one-dimensional). In order to handle the algebra involved, however, we shall consider below a variety of other choices (generally larger) of $\mathscr{H}$ and finite-dimensional $\mathscr{V}$. In superfield terms, the finite-dimensional representations correspond to constrained superfields (Dondi and Jarvis 1981).

The generators $M_{AB} = -[AB]M_{BA}$ of OSp(m/n) satisfy the superalgebra (Jarvis and Green 1979)

$$[M_{AN}, M_{CD}] = g_{BC}M_{AD} - [AB]g_{AC}M_{BD} - [CD]g_{BD}M_{AC} + [AB][CD]g_{AD}M_{BC} \qquad (4)$$

where $1 \leqslant A, B, \ldots \leqslant m+n$, $g_{AB} = [AB]g_{BA}$ is the orthosymplectic metric and the sign factors $[AB]$, $[CD]$, etc, are -1 if $1 \leqslant A \leqslant m$, $m+1 \leqslant B \leqslant m+n$ or *vice versa*, and $+1$ otherwise. Conversion from the M_{AB} basis to the canonical Cartan form (Kac 1978) is readily accomplished and is indicated below in specific examples.

The well known cases of OSp(1/2) and OSp(2/2) are reviewed in § 2 below as an illustration of our method. The main new results come in §§ 3 and 4 with a discussion of OSp(3/2) and OSp(4/2) respectively. The classes of irreducible representations which emerge depend upon the little group $\mathscr{H}_0$. As pointed out above, the need to limit algebraic complexity prohibits the optimal choice of the maximal Abelian subgroup (where $\mathscr{H}$ is a Borel subalgebra: Kac 1978). The little groups for OSp(3/2) and OSp(4/2) are chosen as U(1)×OSp(1/2) and U(1)×SU(2)×SU(2), with corresponding coset spaces of dimension $3 \equiv 1/2$ and $5 \equiv 1/4$, respectively. General induced representations (superfields of arbitrary spin) are constructed using spin projection operators, in the former case extended to matrices for arbitrary 'superspin' irreducible representations of OSp(1/2), derived in § 2 (see also appendices 1 and 2).

In tables 1 and 3 are set out, for the irreducible representations of OSp(3/2) and OSp(4/2) emerging from our technique, the constituents with respect to O(3)×Sp(2) and O(4)×Sp(2), which are isomorphic to SU(2)×SU(2) and SU(2)×SU(2)×SU(2), respectively. They are in general typical (with even and odd dimensions the same). For particular choices of the representation labels, however, atypical representations may arise: for example, for OSp(3/2) the choice $(L=1, M=0)$ yields the fundamental $\mathbf{5} = \mathbf{3} \times \mathbf{1}/\mathbf{1} \times \mathbf{2}$, while in the OSp(4/2) case the choice $(L=1, M=N=0)$ yields the adjoint $\mathbf{17} = (\mathbf{3}\times\mathbf{1}\times\mathbf{1} + \mathbf{1}\times\mathbf{3}\times\mathbf{1} + \mathbf{1}\times\mathbf{1}\times\mathbf{3})/\mathbf{2}\times\mathbf{2}\times\mathbf{2}$. The special cases for OSp(3/2) are in fact $(L=2M+1, M \geqslant \frac{1}{2})$, $(L=1, M \geqslant 0)$ and $(L \geqslant 1, M=0)$ (see table 2 and § 3). For OSp(4/2) the cases $(L=2N+1, M=0, N)$ and $(L=2N+2, M=0, N)$ are treated in § 4; from Kac (1978) more general atypicality conditions exist than these choices, but we have not worked out their general decompositions.

Finally, the connection with Young diagrams has not been made explicit, although some cases are evident. For example, in OSp(3/2) the irreducible representations with $M=0$ and $L=1, 2, 3, \ldots$ (dimensions $\mathbf{5}, \mathbf{12}, \mathbf{20}, \ldots 4(2L-1)$) correspond to the

Table 1. O(3)×Sp(2)≃SU(2)×SU(2) decomposition of typical OSp(3/2) induced representations from little group U(1)×OSp(1/2) for $L \geqslant 3/2$, $M \geqslant 0$ (superfield components given in (17) and (19); see table 2 for special cases).

'Even'	Dimension	'Odd'	Dimension
$A(L, M)$	$(2L+1)(2M+1)$	$a(L, M-\frac{1}{2})$	$(2L+1)(2M)$
$\tilde{H}(L-2, M)$	$(2L-3)(2M+1)$	$\tilde{h}(L-2, M-\frac{1}{2})$	$(2L-3)(2M)$
$\tilde{P}^0(L-1, M)$	$(2L-1)(2M+1)$	$\tilde{\psi}^-(L-1, M-\frac{1}{2})$	$(2L-1)(2M)$
$P^{-1}(L-1, M-1)$	$(2L-1)(2M-1)$	$\psi^+(L-1, M+\frac{1}{2})$	$(2L-1)(2M+2)$
Total	$2(2L-1)(4M+1)$	Total	$2(2L-1)(4M+1)$

Table 2. $O(3)\times Sp(2)\simeq SU(2)\times SU(2)$ decompositions of OSp(3/2) induced representations from little group $U(1)\times OSp(1/2)$ for special cases (atypical representations).

$(L=2M+1, M\geqslant 1)$ invariant space†			
$P^{-1}(2M, M-1)$	$(4M+1)(2M-1)$	$\psi^{-}(2M, M-\frac{1}{2})$	$(4M+1)(2M)$
$\tilde{H}(2M-1, M)$	$(4M-1)(2M+1)$	$\tilde{h}(2M-1, M-\frac{1}{2})$	$(4M-1)(2M)$
Total	$16M^2-2$	Total	$16M^2$
$(L=2M+1, M\geqslant\frac{1}{2})$ factor space			
$A(2M+1, M)$	$(2M+1)(4M+3)$	$a(2M+1, M-\frac{1}{2})$	$(4M+3)(2M)$
$\tilde{P}^0(2M, M)$	$(2M+1)(4M+1)$	$\psi^{+}(2M, M+\frac{1}{2})$	$(4M+1)(2M+2)$
Total	$16(M+\frac{1}{2})^2$	Total	$16(M+\frac{1}{2})^2-2$
$(L=1, M\geqslant 0)$ invariant space			
$A(1, M)$	$3(2M+1)$	$\psi^{+}(0, M+\frac{1}{2})$	$1(2M+2)$
$P^{-1}(0, M-1)$	$1(2M-1)$	$a^{-}(1, M-\frac{1}{2})$	$3(2M)$
Total	$8M+2$	Total	$8M+2$
$(L\geqslant 1, M=0)$ invariant space			
$A(L, 0)$	$2L+1$	$\psi^{+}(L-1, \frac{1}{2})$	$2(2L-1)$
$\tilde{H}(L-2, 0)$	$2L-3$		
Total	$4L-2$		$4L-2$

† For $M=\frac{1}{2}$, $\tilde{H}$ and $\tilde{\psi}^{-}$ themselves form an invariant subspace (equivalent to the fundamental **5**).

Table 3. $O(4)\times Sp(2)\simeq SU(2)\times SU(2)\times SU(2)$ decomposition of typical OSp(4/2) irreducible representations from little group $U(1)\times SU(2)\times SU(2)$ for $L\geqslant\frac{3}{2}$, $M, N\geqslant 0$ (superfield components given in (21) and (23), see text for special cases).

'Even'	Dimension	'Odd'	Dimension
$A, \tilde{D}(L-1\pm 1, M, N)$	$2(2L-1)(2M+1)(2N+1)$	$\psi^{+\pm}(L-\frac{1}{2}, M+\frac{1}{2}, N\pm\frac{1}{2})$	$2(2L)(2M+2)(2N+1)$
$\tilde{F}^0(L-1, M, N)$	$(2L-1)(2M+1)(2N+1)$	$\psi^{-\pm}(L-\frac{1}{2}, M-\frac{1}{2}, N\pm\frac{1}{2})$	$2(2L)(2M)(2N+1)$
$F^{\pm}(L-1, M\pm 1, N)$	$2(2L-1)(2M+1)(2N+1)$	$\tilde{\chi}^{+\pm}(L-\frac{3}{2}, M+\frac{1}{2}, N\pm\frac{1}{2})$	$2(2L-2)(2M+2)(2N+1)$
$\tilde{G}^0(L-1, M, N)$	$(2L-1)(2M+1)(2N+1)$	$\tilde{\chi}^{-\pm}(L-\frac{3}{2}, M-\frac{1}{2}, N\pm\frac{1}{2})$	$2(2L-2)(2M)(2N+1)$
$G^{\pm}(L-1, M, N\pm 1)$	$2(2L-1)(2M+1)(2N+1)$		
Total	$8(2L-1)(2M+1)(2N+1)$	Total	$8(2L-1)(2M+1)(2N+1)$

totally graded-symmetrical traceless tensors (where the $\mathbf{3}\times\mathbf{1}$ constituent of the fundamental **5** is chosen to be even). Extensions of our techniques to cover these and other points are for future work.

2. Illustrative examples: OSp(1/2) and OSp(2/2)

To illustrate the techniques used to obtain the representations of OSp(3/2) and OSp(4/2) in §§ 3 and 4 respectively, we consider here in some detail the method as applied to the simpler cases of OSp(1/2) and OSp(2/2). More specifically we use OSp(1/2) to demonstrate the procedure for finding finite-dimensional representations. The OSp(2/2) case we use to introduce the matrices $\hat{\mathcal{M}}_{\alpha\beta}$ and $\hat{\mathcal{Q}}_\alpha$ (see appendix 2) which are needed to find the general spin-M representations for OSp(3/2).

2.1. OSp(1/2)

The OSp(1/2) superalgebra consists of the even generators $M_{\alpha\beta}$ and the odd generators $M_{1\alpha}$ where $1 \leqslant \alpha, \beta \leqslant 2$. If we call $M_{1\alpha} \equiv Q_\alpha$ and transform $M_{\alpha\beta}$ to the spherical basis M_+, M_- and M_3 via $M_{\alpha\beta} = 2\boldsymbol{M} \cdot (\boldsymbol{\sigma}\varepsilon)_{\alpha\beta}$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices and $(\varepsilon_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, or $M_+ = \frac{1}{2}M_{22}$, $M_- = -\frac{1}{2}M_{11}$ and $M_3 = \frac{1}{2}M_{12}$ we obtain the following superalgebra:

$$
\begin{aligned}
&[M_3, Q_\alpha] = -\tfrac{1}{2}(\sigma_3)_\alpha^\beta Q_\beta \qquad && [M_\pm, Q_\alpha] = -(\sigma_\pm)_\alpha^\beta Q_\beta \\
&[M_+, M_-] = 2M_3 && [M_3, M_\pm] = \pm M_\pm \\
&\{Q_\alpha, Q_\beta\} = -2(\sigma_+\varepsilon)_{\alpha\beta}M_- - 2(\sigma_-\varepsilon)_{\alpha\beta}M_+ - 2(\sigma_3\varepsilon)_{\alpha\beta}M_3
\end{aligned}
\tag{5}
$$

with all other (anti)commutators zero.

For the subalgebra $\mathcal{H}$ we choose $\mathcal{H} = \{M_3, M_+, Q_2\}$ with $\mathcal{H}_0 = \{M_3\} \simeq \mathrm{U}(1)$. The cosets are labelled by the elements $\exp(xM_- + \theta Q_1)$ and the superfields are functions $\Phi(x, \theta)$ carrying a charge $\hat{M} \equiv -M$. Expanding the superfield in θ we have simply $\Phi(x, \theta) = A(x) + \theta\psi(x)$.

The differential representation of the generators (see (3)) is

$$
\begin{aligned}
M_- &= \partial/\partial x \\
M_+ &= -x^2\,\partial/\partial x - x\theta\,\partial/\partial\theta + 2xM \\
M_3 &= -x\,\partial/\partial x - \tfrac{1}{2}\theta\,\partial/\partial\theta + M \\
Q_1 &= -\theta\,\partial/\partial x + \partial/\partial\theta \\
Q_2 &= -\theta x\,\partial/\partial x + x\,\partial/\partial\theta + 2\theta M.
\end{aligned}
\tag{6}
$$

Acting on the superfield with the above set of generators we obtain the following variations for the component fields, writing $A' \equiv \partial A/\partial x$, etc,

$$
\begin{aligned}
&M_-\colon && \delta A = A' && \delta\psi = \psi' \\
&M_+\colon && \delta A = -x^2A' - 2MxA && \delta\psi = -x^2\psi' - x\psi - 2Mx\psi \\
&M_3\colon && \delta A = -xA' - MA && \delta\psi = -x\psi' - \tfrac{1}{2}\psi - M\psi \\
&Q_1\colon && \delta A = \psi && \delta\psi = -A' \\
&Q_2\colon && \delta A = x\psi && \delta\psi = -xA' - 2MA.
\end{aligned}
\tag{7}
$$

We now expand $A(x)$ and $\psi(x)$ as power series in x:

$$
A(x) = \sum_{n=0}^{\infty} A^n x^n \qquad \text{and} \qquad \psi(x) = \sum_{n=0}^{\infty} \psi^n x^n.
$$

Substituting for these into (7) and equating like powers of x, we obtain the results

$$
\begin{array}{lll}
M_-: & \delta A^n = (n+1)A^{n+1} & \delta\psi^n = (n+1)\psi^{n+1} \\
M_+: & \delta A^n = -(n-1-2M)A^{n-1}, n \geq 1 & \delta\psi^n = -(n-2M)\psi^{n-1}, n \geq 1 \\
M_3: & \delta A^n = -(n-M)A^n & \delta\psi^n = -(n+\tfrac{1}{2}-M)\psi^n \\
Q_1: & \delta A^n = \psi^n, n \leq 2M-1 & \delta\psi^n = -(n+1)A^{n+1} \\
Q_2: & \delta A^n = \psi^{n-1}, n \geq 1 & \delta\psi^n = -(n-2M)A^{n-1}, n \geq 1
\end{array} \tag{8}
$$

with all other variations zero.

If we take M half-integral, then it is clear from the explicit component form of the variations, (8), especially M_+ and Q, that the infinite set $\{A^0, A^1, \ldots; \psi^0, \psi^1, \ldots\}$ has an infinite invariant subset $\{A^{2M+1}, A^{2M+2}, \ldots; \psi^{2M}, \psi^{2M+1}, \ldots\}$. If these components are set to zero, then the remaining finite subset $\{A^0, A^1, \ldots, A^{2M}; \psi^0, \psi^1, \ldots, \psi^{2M-1}\}$ is invariant (i.e. as a factor space).

Thus an arbitrary finite-dimensional irreducible representation of OSp(1/2) has dimension $4M+1$ and 'superspin' M: i.e. spins M and $M-\frac{1}{2}$ under Sp(2) (see, for example, Scheunert *et al* 1977). The matrix elements acquire a more symmetrical form in the basis defined by $B^\mu = A^{\mu+M}$, $\chi^\nu = \psi^{\nu+M-\frac{1}{2}}$ or $\mu = -M, -M+1, \ldots, M$ and $\nu = -M+\frac{1}{2}, -M+\frac{3}{2}, \ldots, M-\frac{1}{2}$:

$$
\begin{array}{lll}
M_3: & \delta B^\mu = -\mu B^\mu & \delta\chi^\nu = -\nu\chi^\nu \\
M_+, M_-: & \delta B^\mu = (M+1\mp\mu)B^{\mu\mp1} & \delta\psi^\nu = (M+\tfrac{1}{2}\mp\nu)\chi^{\nu\mp1} \\
Q_1, Q_2: & \delta B^\mu = \chi^{\mu\pm\frac{1}{2}} & \delta\chi^\nu = -(\nu\pm M\pm\tfrac{1}{2})B^{\nu-\frac{1}{2}\pm1}
\end{array} \tag{9}
$$

where $B^{\pm(M+1)} \equiv \chi^{\pm(M+\frac{1}{2})} \equiv 0$. An alternative form for these matrices is given in appendix 2 in terms of spin projection operators (see, for example, (A6) and (A9)); it is in this form that they are required in the OSp(2/2) and OSp(3/2) cases.

2.2. *OSp(2/2)*

The OSp(2/2) superalgebra consists of the odd generators $Q_{a\alpha} \equiv M_{a\alpha}$, the O(2) generator $L \equiv M_{12}$ and the Sp(2) generators $M_{\alpha\beta}$. Here $1 \leq a, b \leq 2$ refers to O(2) and $1 \leq \alpha, \beta \leq 2$ refers to Sp(2). These generators satisfy the superalgebra

$$
\begin{aligned}
&[L, Q_{1\alpha}] = -Q_{2\alpha} \qquad [L, Q_{2\alpha}] = Q_{1\alpha} \\
&[M_{\alpha\beta}, M_{\gamma\delta}] = \varepsilon_{\alpha\gamma}M_{\beta\delta} + \varepsilon_{\alpha\delta}M_{\beta\gamma} + \varepsilon_{\beta\gamma}M_{\alpha\delta} + \varepsilon_{\beta\delta}M_{\alpha\gamma} \\
&[M_{\alpha\beta}, Q_{a\gamma}] = \varepsilon_{\beta\gamma}Q_{a\alpha} + \varepsilon_{\alpha\gamma}Q_{a\beta} \\
&\{Q_{a\alpha}, Q_{b\beta}\} = -\delta_{ab}M_{\alpha\beta} - \varepsilon_{\alpha\beta}L_{ab}
\end{aligned} \tag{10}
$$

with all other (anti)commutators zero.

For the subalgebras $\mathscr{H}$ and $\mathscr{H}_0$ we choose $\mathscr{H} = \mathscr{H}_0 = \{M_{\alpha\beta}, Q_{1\alpha}\} \simeq \text{OSp}(1/2)$. Although this is obviously not the simplest choice for $\mathscr{H}_0$, we nevertheless make this choice to demonstrate the use of the OSp(1/2) little group which arises also in the study of OSp(3/2). The cosets are labelled by the elements $\exp(xL + \theta^\alpha Q_{2\alpha})$. Superfields are functions $\Phi_A(x, \theta^\alpha)$ and form a 'superspin' M representation of

OSp(1/2) as described above (see also (A6)):

$$\Phi_A(x, \theta_\beta) = \begin{pmatrix} \phi_a(x, \theta_\beta) \\ \phi_{a\alpha}(x, \theta_\beta) \end{pmatrix}.$$

In the following work all spin-M indices will be suppressed. Expanding the superfield in θ we obtain the general forms

$$\begin{pmatrix} \phi \\ \phi_\alpha \end{pmatrix} = \begin{pmatrix} A \\ a_\alpha \end{pmatrix} + \theta^\beta \begin{pmatrix} \psi_\beta^+ + \psi_\beta^- \\ P_\beta \end{pmatrix} + \tfrac{1}{2}\theta^2 \begin{pmatrix} H \\ h_\alpha \end{pmatrix} \tag{11}$$

where $\theta^2 \equiv \varepsilon_{\alpha\beta}\theta^\alpha\theta^\beta$.

The components have the following spins: A and H, M; ψ_α^+, $M+\frac{1}{2}$; ψ_α^-, a_α and h_α, $M-\frac{1}{2}$. To decompose $P_{\alpha\beta}$ into fields of definite spin we proceed as follows:

$$\begin{aligned} P_{\alpha\beta} &= \tfrac{1}{2}(P_{\alpha\beta} - P_{\beta\alpha}) + \tfrac{1}{2}(P_{\alpha\beta} + P_{\beta\alpha}) \\ &= \tfrac{1}{2}\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta}P_{\delta\gamma} + P^0_{\alpha\beta} + P^{-1}_{\alpha\beta} \end{aligned}$$

where

$$P^{0,-1}_{\alpha\beta} = \Pi^{0,-1\gamma\delta}_{\alpha\beta}\tfrac{1}{2}(P_{\gamma\delta} + P_{\delta\gamma})$$

are the spin-M and spin-$(M-1)$ projections defined in appendix 1. Since $\theta^\beta P_{\beta\alpha}$ has spin $M-\frac{1}{2}$ but $\Pi_\alpha^{-\frac{1}{2}\beta}\,\Pi_{\beta\gamma}^{+1\delta\varepsilon} = 0$, there is no spin-$(M+1)$ projection. Furthermore, using (A1) and (A5),

$$\varepsilon^{\gamma\delta}P_{\delta\gamma} = \hat{M}^{\gamma\delta}P_{\gamma\delta}/2(M+1) \equiv \hat{M}^{\gamma\delta}P^0_{\gamma\delta}/2(M+1).$$

It should be noted that $P^0_{\alpha\beta}$ is not an eigenvector of $\Pi_\gamma^{\pm\frac{1}{2}\delta}$. We can, however, rewrite it as

$$(2M+1)P^0_{\alpha\beta} = M(P^0_+)_{\alpha\beta} + (M+1)(P^0_-)_{\alpha\beta}$$

where

$$\begin{aligned} &(P^0_\pm)_{\alpha\beta} = (P^0_{\alpha\beta} - \varepsilon_{\alpha\beta}\hat{M}^{\gamma\delta}P^0_{\gamma\delta}/2M^\pm) \\ &M^+ \equiv M \qquad\qquad M^- \equiv -M-1 \end{aligned}$$

such that $\Pi_\alpha^{\pm\frac{1}{2}\beta}(P^0_\pm)_{\gamma\beta} = (P^0_\pm)_{\gamma\alpha}$ and $\Pi_\alpha^{\pm\frac{1}{2}\beta}(P^0_\mp)_{\gamma\beta} = 0$. Thus finally we have in (11)

$$P_{\alpha\beta} = (P^0_-)_{\alpha\beta} + P^{-1}_{\alpha\beta}. \tag{12}$$

The differential representation of the generators (see (3)), writing $\partial_\alpha \equiv \partial/\partial\theta^\alpha$, is

$$\begin{aligned} M_{\alpha\beta} &= \theta_\alpha\,\partial_\beta + \theta_\beta\,\partial_\alpha - \hat{\mathcal{M}}_{\alpha\beta} \\ Q_{1\alpha} &= \theta_\alpha\,\partial/\partial x + x\,\partial_\alpha - \hat{\mathcal{Q}}_{1\alpha} \\ Q_{2\alpha} &= f\,\partial_\alpha + \tfrac{1}{2}\theta^2 f^{-1}\,\partial_\alpha \\ L &= f\,\partial/\partial x + \tfrac{1}{2}\theta^2 f^{-1}\,\partial/\partial x \end{aligned} \tag{13}$$

where $f(x) = (K - x^2)^{1/2}$ and K is an undetermined constant†. $\hat{\mathcal{Q}}_{1\alpha}$ and $\hat{\mathcal{M}}_{\alpha\beta}$ are matrices in the 'superspin' M representation of OSp(1/2) as discussed in appendix 2 (see (A7) and (A8)).

† In this case arbitrary powers of x arise from brackets such as $[xL, [xL, \ldots [xL, \eta Q_{1\alpha}] \ldots]$. Here K arises as an integration constant in the differential equations satisfied by $f(x)$. We find that if $\hat{\mathcal{M}}_{\alpha\beta}$ and $\hat{\mathcal{Q}}_{1\alpha}$ are to enter $M_{\alpha\beta}$ and $Q_{1\alpha}$ in the manner suggested by the action of the little group on the cosets, then they are not present in $Q_{2\alpha}$ and L.

Examining first the action of L we find the following results for the variations of the components:

$$
\begin{aligned}
&\delta A = fA' && \delta a_\alpha = fa'_\alpha \\
&\delta\psi^+_\alpha = f\psi'^+_\alpha && \delta P^{-1}_{\alpha\beta} = fP'^{-1}_{\alpha\beta} \\
&\delta\psi^-_\alpha = f\psi'^-_\alpha && \delta P^0_{\alpha\beta} = fP'^0_{\alpha\beta} \\
&\delta H = fH' + f^{-1}A' && \delta g_\alpha = fh'_\alpha + f^{-1}a'_\alpha
\end{aligned}
\tag{14}
$$

To more readily extract the finite-dimensional representations (cf (7) and (8)), we replace H and h_α by $\tilde{H}$ and $\tilde{h}_\alpha$ such that, under L,

$$\delta\tilde{H} = f(\tilde{H}' + ex^{-1}\tilde{H}) \qquad \text{and} \qquad \delta\tilde{h}_\alpha = f(\tilde{h}'_\alpha + dx^{-1}\tilde{h}_\alpha).$$

We find that

$$\tilde{H} = x^{-e}H + x^{1-e}A' \qquad \text{and} \qquad \tilde{h}_\alpha = x^{-d}h_\alpha + x^{1-d}a'_\alpha \tag{15}$$

satisfy these criteria (with the choice $d = e = 0$, for example). Using (15), the variations of the component fields (11) under the action of, for example, the generator $Q_{1\alpha}$ are (here $[M] \equiv (2M+1)^{1/2}$)

$$
\begin{aligned}
\delta A &= x\psi^+_\alpha + x\psi^-_\alpha - [M]a_\alpha \\
\delta\psi^+_\beta &= -(\Pi^{+\frac{1}{2}}\varepsilon)_{\beta\alpha}(f^2A' + Kx^{e+1}\tilde{H})/K + 2[M]^{-1}(P^0_+)_{\alpha\beta} \\
\delta\psi^-_\beta &= -(\Pi^{-\frac{1}{2}}\varepsilon)_{\beta\alpha}(f^2A' + Kx^{e+1}\tilde{H})/K + [M]^{-1}(P^0_-)_{\alpha\beta} + [M]P^{-1}_{\alpha\beta} \\
\delta\tilde{H} &= -x^{-e}(f^2\psi'^+_\alpha - x\psi^+_\alpha + f^2\psi'^-_\alpha - x\psi^-_\alpha)/K - [M]x^{-e}(a_\alpha + Kx^d h_\alpha)/K \\
\delta a_\beta &= x(P^0_-)_{\alpha\beta} + xP^{-1}_{\alpha\beta} - [M](\Pi^{-\frac{1}{2}}\varepsilon)_{\beta\alpha}A \\
\delta P^0_{\beta\gamma} &= \hat{M}_{\beta\gamma}[-(f^2a'_\alpha + Kx^{1+d}\tilde{h}_\alpha)/K + [M]\psi^-_\alpha - 2M\psi^+_\alpha]/2M \\
\delta P^{-1}_{\beta\gamma} &= (\Pi^{-1})^{\delta\varepsilon}_{\beta\gamma}\varepsilon_{\alpha\delta}[(f^2a'_\varepsilon + Kx^{1+d}\tilde{h}_\varepsilon)/K - [M]\psi^-_\varepsilon] \\
\delta\tilde{h}_\beta &= -x^{-d}(f^2(P'^0_-)_{\alpha\beta} - x(P^0_-)_{\alpha\beta} + f^2P'^{-1}_{\alpha\beta} - xP^{-1}_{\alpha\beta})/K \\
&\qquad - [M]x^{-d}(\Pi^{-\frac{1}{2}}\varepsilon)_{\beta\alpha}(A + Kx^e\tilde{H})/K.
\end{aligned}
$$

similar matrix elements are found for the remaining generators $Q_{2\alpha}$ and $M_{\alpha\beta}$.

We shall not go on to extract the finite-dimensional representations (invariant factors) as OSp(2/2) has been treated in the literature under SU(2/1) (to which it is ismorphic) (see, for example, Scheunert *et al* 1977, Marcu 1980a, b, and references therein). The above choice of basis was made mainly for purposes of illustration; a comprehensive superfield treatment of SU(2/1) and SU(n/1) has been given by Dondi and Jarvis (1981) in a more convenient basis. However, with the superfield techniques at hand, we now proceed in the next sections to examine the cases of OSp(3/2) and OSp(4/2).

3. Osp (3/2)

The OSp(3/2) superalgebra consists of the odd generators $Q_{a\alpha} \equiv M_{a\alpha}$, the O(3) generators $L_{ab} \equiv M_{ab}$ and the Sp(2) generators $M_{\alpha\beta}$. Here $1 \leqslant a, b \leqslant 3$ refer to O(3) and

$1 \leqslant \alpha, \beta \leqslant 2$ refer to Sp(2). We recast these generators into the form

$$L_+ = L_{31} + \mathrm{i}L_{32} \qquad L_- = -L_{31} + \mathrm{i}L_{32} \qquad L_3 = \mathrm{i}L_{21}$$
$$Q_{+\alpha} = Q_{1\alpha} + \mathrm{i}Q_{2\alpha} \qquad Q_{-\alpha} = Q_{1\alpha} - \mathrm{i}Q_{2\alpha} \qquad Q_{3\alpha} = Q_{3\alpha}.$$

These generators satisfy the following algebra:

$$\begin{aligned}
&[L_+, L_-] = 2L_3 \qquad [L_3, L_\pm] = \pm L_\pm \\
&[M_{\alpha\beta}, M_{\gamma\delta}] = \varepsilon_{\beta\gamma}M_{\alpha\delta} + \varepsilon_{\alpha\delta}M_{\beta\gamma} + \varepsilon_{\alpha\gamma}M_{\beta\delta} + \varepsilon_{\beta\delta}M_{\alpha\gamma} \\
&[L_\pm, Q_{\mp\alpha}] = \pm 2Q_{3\alpha} \qquad [L_\pm, Q_{3\alpha}] = \mp Q_{\pm\alpha} \\
&[L_3, Q_{\pm\alpha}] = \pm Q_{\pm\alpha} \\
&[M_{\alpha\beta}, Q_{\pm\gamma}] = \varepsilon_{\alpha\gamma}Q_{\pm\beta} + \varepsilon_{\beta\gamma}Q_{\pm\alpha} \\
&[M_{\alpha\beta}, Q_{3\gamma}] = \varepsilon_{\alpha\gamma}Q_{3\beta} + \varepsilon_{\beta\gamma}Q_{3\alpha} \\
&\{Q_{3\alpha}, Q_{\pm\beta}\} = \mp\varepsilon_{\alpha\beta}L_\pm \qquad \{Q_{3\alpha}, Q_{3\beta}\} = -M_{\alpha\beta} \\
&\{Q_{+\alpha}, Q_{-\beta}\} = -2M_{\alpha\beta} - 2\varepsilon_{\alpha\beta}L_3
\end{aligned} \tag{16}$$

with all other (anti)commutators zero. For the subalgebra $\mathscr{H}$ we choose

$$\mathscr{H} = \{L_+, L_3, Q_{+\alpha}, Q_{3\alpha}, M_{\alpha\beta}\}$$

and

$$\mathscr{H}_0 = \{L_3, Q_{3\alpha}, M_{\alpha\beta}\} \simeq \mathrm{U}(1) \times \mathrm{Osp}(1/2).$$

Cosets are labelled by the elements $\exp(xL_- + \theta^\alpha Q_{-\alpha})$ and superfields are functions $\Phi(x, \theta_\alpha)$ carrying charge $\hat{L} \equiv -L$, and a 'superspin' M representation of the $\mathrm{U}(1)\times$ OSp(1/2) little group. The superfield expanded in θ takes the following form as described in § 2 (cf (11) and (12)):

$$\Phi(x, \theta) = \begin{pmatrix} \phi(x, \theta) \\ \phi_\alpha(x, \theta) \end{pmatrix} = \begin{pmatrix} A \\ a_\alpha \end{pmatrix} + \theta^\beta \begin{pmatrix} \psi^+_\beta + \psi^-_\beta \\ (P^0_-)_{\beta\alpha} + P^{-1}_{\beta\alpha} \end{pmatrix} + \tfrac{1}{2}\theta^2 \begin{pmatrix} H \\ h_\alpha \end{pmatrix}. \tag{17}$$

The differential representation of the generators is from (3) (see also (13) and appendix 2):

$$\begin{aligned}
&L_- = \partial/\partial x \\
&L_+ = -x^2\, \partial/\partial x - \theta^2\, \partial/\partial x - 2x\theta^\alpha\, \partial_\alpha + 2xL - 2\theta^\alpha \hat{\mathscr{Q}}_{3\alpha} \\
&L_3 = -x\, \partial/\partial x - \theta^\alpha\, \partial_\alpha + L \\
&M_{\alpha\beta} = \theta_\alpha\, \partial_\beta + \theta_\beta\, \partial_\alpha - \hat{\mathscr{M}}_{\alpha\beta} \\
&Q_{-\alpha} = \delta_\alpha \\
&Q_{+\alpha} = -2\theta_\alpha x\, \partial/\partial x - x^2\theta_\alpha + 2\theta^2\, \partial_\alpha + 2\theta_\alpha L - 2\theta^\beta \hat{\mathscr{M}}_{\beta\alpha} - 2x\hat{\mathscr{Q}}_{3\alpha} \\
&Q_{3\alpha} = -\theta_\alpha\, \partial/\partial x - x\, \partial_\alpha - \hat{\mathscr{Q}}_{3\alpha}.
\end{aligned} \tag{18}$$

As shown in § 2 we examine the action of $L_\pm$ to obtain a modified basis for the component fields in which the finite-dimensional factor is most readily seen. The definitions of these component fields and their degrees (highest power of x in the finite factor) are (where $[M]=(2M+1)^{1/2}$): A, a, $2L$; ψ^+, $\tilde{\psi}^-$, $\tilde{P}^0_{\alpha\beta}$ and $P^{-1}_{\alpha\beta}$, $2L-2$; $\tilde{H}$, $\tilde{h}$, $2L-4$, where

$$\begin{aligned}
\tilde{\psi}^-_\alpha &= L[M]^{-1}\psi^-_\alpha + a'_\alpha \\
\tilde{P}^0_{\alpha\beta} &= -L[M]P^0_{\alpha\beta} + \tfrac{1}{2}\hat{M}_{\alpha\beta}A' \\
\tilde{H} &= (M+1)[M]^{-1}(L-1)H - \tfrac{1}{2}L^{-1}[M]^{-1}\hat{M}^{\alpha\beta}\tilde{P}^{0\prime}_{\alpha\beta} \\
&\qquad + [M]^{-1}L^{-1}(2L-1)^{-1}(L-1)(M+1)(L+2M)A'' \\
\tilde{h}_\alpha &= (L-1)[M]^{-1}h_\alpha + [M]L^{-1}\tilde{\psi}'^-_\alpha + [M]^{-1}L^{-1}(2L-1)^{-1}(L-1)(L-2M-1)a''_\alpha.
\end{aligned} \tag{19}$$

From (18), taking into account the definitions (19), the $O(3)\times Sp(2) \simeq SU(2)\times SU(2)$ decompositions we obtain for arbitrary induced representations with the chosen little group (corresponding to superfields of arbitrary half-integer charge L and 'superspin' M) are given in table 1. This class of irreducible representations is in general typical (with even and odd dimensions the same), and total dimension $4(2L-1)(4M+1)$ with $L\geq\frac{3}{2}$ and $M\geq 0$. In the basis (19) it is found that superfields which cannot be decomposed arise for certain (L, M) values, corresponding to atypical representations. With $(L=2M+1, M\geq\frac{1}{2})$ (cf Kac 1978) the set $P^{-1}_{\alpha\beta}, \tilde{H}, \tilde{\psi}^-_\alpha$ and $\tilde{h}_\alpha$ form an invariant subspace of dimension $32M^2-2=16M^2-2/16M^2$, with the set A, $\tilde{P}^0_{\alpha\beta}$, a_α and ψ^+_α invariant as a factor space (for $M=\frac{1}{2}$, $\tilde{H}$ and $\tilde{\psi}^-_\alpha$ form a further invariant subspace equivalent to the fundamental **5**). From (19) it is also evident that $L=0$, $\frac{1}{2}$ and 1 are special cases; $(L=0, M=0)$ is a singlet, but no finite-dimensional $(L=\frac{1}{2}, M\geq 0)$ superfield can be constructed; thirdly, the sequence $(L=1, M\geq 0)$, with invariant set A, $P^-_{\alpha\beta}$, ψ^+_α and a^-_α, includes the fundamental $\mathbf{5}=\mathbf{3}\times\mathbf{1}/\mathbf{1}\times\mathbf{2}$ for $M=0$, and the adjoint $\mathbf{12}=(\mathbf{3}\times\mathbf{1}+\mathbf{1}\times\mathbf{3})/\mathbf{3}\times\mathbf{2}$ for $M=\frac{1}{2}$. The $O(3)\times Sp(2)\simeq SU(2)\times SU(2)$ decompositions obtained for these cases are summarised in table 2. Finally, that there is some connection with the graded Young diagrams (Dondi and Jarvis 1981) is seen already from the fact that the series $L=1, 2, 3, \ldots$ with $M=0$ (with dimensions **5**, **12**, **20**, ...) correspond to the totally graded-symmetrical traceless tensors of rank L (where the $\mathbf{3}\times\mathbf{1}$ constituent of the fundamental **5** is chosen to be even).

4. *OSp (4/2)*

The OSp(4/2) superalgebra can be cast in a more useful form as follows. Let $0\leq\mu, \nu\leq 3$ for the even indices and $1\leq\alpha,\beta\leq 2$ for the odd indices, and from (4) define†

$$\begin{aligned}
L_{\dot a\dot b} &= -\tfrac{1}{2}(\bar{\sigma}^{\mu\nu})_{\dot a\dot b}M_{\mu\nu} \\
M_{ab} &= \tfrac{1}{2}(\sigma^{\mu\nu})_{ab}M_{\mu\nu} \\
N_{\alpha\beta} &\equiv M_{\alpha\beta} \\
Q_{\dot a a\alpha} &= \sigma^{\mu}_{a\dot a}M_{\mu\alpha}
\end{aligned}$$

where

$$\sigma_\mu = (1, \boldsymbol{\sigma}) \qquad \bar{\sigma}_\mu = (1, -\boldsymbol{\sigma}) \qquad \sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu = 2\eta_{\mu\nu}$$

$$\sigma_{\mu\nu} = \tfrac{1}{2}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu) \qquad \bar{\sigma}_{\mu\nu} = \tfrac{1}{2}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu).$$

† The same complex algebra results also from a Euclidean choice of metric.

These generators satisfy the superalgebra

$$\begin{aligned}
&\{Q_{\dot{a}a\alpha}, Q_{\dot{b}b\beta}\} = -2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}N_{\alpha\beta} + \varepsilon_{ab}\varepsilon_{\alpha\beta}L_{\dot{a}\dot{b}} + \varepsilon_{\alpha\beta}\varepsilon_{\dot{a}\dot{b}}M_{ab}\\
&[M_{ab}, M_{cd}] = \varepsilon_{bc}M_{ad} + \varepsilon_{ad}M_{bc} + \varepsilon_{ac}M_{bd} + \varepsilon_{bd}M_{ac}\\
&[M_{ab}, Q_{\dot{c}c\gamma}] = \varepsilon_{bc}Q_{\dot{c}a\gamma} + \varepsilon_{ac}Q_{\dot{c}b\gamma}
\end{aligned} \tag{20}$$

and similarly for $L_{\dot{a}\dot{b}}$ and $N_{\alpha\beta}$, corresponding to the special case $\alpha = 1$ of the $D(2, 1, \alpha)$ exceptional superalgebra (Kac 1978, Rittenberg 1978, Parker 1980). Corresponding to $L_{\dot{a}\dot{b}}$, generators L_+, L_- and L_3 can be defined as in § 2 above. For the subalgebra $\mathscr{H}$ we choose

$$\mathscr{H} = \{L_3, M_{ab}, N_{\alpha\beta}, L_+, Q_{\dot{2}a\alpha}\}$$

and

$$\mathscr{H}_0 = \{L_3, M_{ab}, N_{\alpha\beta}\} \simeq \mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2).$$

Cosets are labelled by the elements $\exp(xL_- + \theta^{a\alpha}Q_{\dot{1}a\alpha})$, and superfields are functions $\Phi(x, \theta_{a\alpha})$ carrying a charge $\hat{L} \equiv -L$ and spins $M\times N$ under the little group $\mathrm{U}(1)\times \mathrm{SU}(2)\times\mathrm{SU}(2)$:

$$\begin{aligned}
\Phi(x, \theta) = {} & A(x) + \theta^{a\alpha}\sum(\psi^{\pm\pm}_{a\alpha}(x) + \psi^{\pm\mp}_{a\alpha}(x)) + (\theta\theta)^{ab}\Big(F^0_{ab}(x) + \sum F^{\pm}_{ab}(x)\Big)\\
& + (\theta\theta)^{\alpha\beta}\Big(G^0_{\alpha\beta}(x) + \sum G^{\pm}_{\alpha\beta}(x)\Big) + (\theta^3)^{a\alpha}\sum(\chi^{\pm\pm}_{a\alpha}(x) + \chi^{\pm\mp}_{a\alpha}(x)) + \theta^4 D(x)
\end{aligned} \tag{21}$$

where the monomial basis for θ expansions is

$$\begin{aligned}
&(\theta\theta)^{ab} = \theta^{a\alpha}\theta^{b}_{\alpha} \qquad && (\theta\theta)^{\alpha\beta} = \theta^{a\alpha}\theta^{\beta}_{a}\\
&(\theta^3)^{a\alpha} = (\theta\theta)^{ab}\theta^{\alpha}_{b} \qquad && \theta^4 = (\theta^3)^{a\alpha}\theta_{a\alpha}
\end{aligned}$$

and the summation is over all possible projections onto total spin $M\pm\frac{1}{2}$, $N\pm\frac{1}{2}$ (for $\psi_{a\alpha}(x)$ and $\chi_{a\alpha}(x)$), $M\pm 1$, $M+0$ and $N\pm 1$, $N+0$ (for $F_{ab}(x)$ and $G_{\alpha\beta}(x)$, respectively). The relevant projection operators and their properties are given in appendix 1. Working with the θ algebra requires a calculus of products such as

$$\begin{aligned}
&(\theta\theta)^{ab}\theta^{c\gamma} = -\tfrac{1}{3}\varepsilon^{bc}(\theta^3)^{a\gamma} - \tfrac{1}{3}\varepsilon^{ac}(\theta^3)^{b\gamma}\\
&(\theta\theta)^{ab}(\theta\theta)^{\gamma\delta} = 0\\
&(\theta\theta)^{ab}(\theta\theta)^{cd} = \tfrac{1}{6}(\varepsilon^{bc}\varepsilon^{ad} + \varepsilon^{ac}\varepsilon^{bd})\theta^4
\end{aligned}$$

which can all be obtained by symmetry arguments.

The differential representation (see (3)) writing $\partial_{a\alpha} \equiv \partial/\partial\theta^{a\alpha}$ is

$$\begin{aligned}
L_- &= \partial/\partial x\\
L_+ &= -(x^2 - \tfrac{1}{2}\theta^4)\,\partial/\partial x - (x\theta + \theta^3)^{a\alpha}\,\partial_{a\alpha} + 2xL + \tfrac{1}{2}(\theta\theta)^{ab}\hat{M}_{ab} - (\theta\theta)^{\alpha\beta}\hat{N}_{\alpha\beta}\\
L_3 &= -(x\,\partial/\partial x + \tfrac{1}{2}\theta^{a\alpha}\,\partial_{a\alpha}) + L\\
Q_{\dot{1}a\alpha} &= \theta_{a\alpha}\,\partial/\partial x + \partial_{a\alpha}\\
Q_{\dot{2}a\alpha} &= (x\theta_{a\alpha} + \theta^3_{a\alpha})\,\partial/\partial x + x\,\partial_{a\alpha} - 2(\theta\theta)^{b}_{a}\,\partial_{b\alpha} + (\theta\theta)^{\beta}_{\alpha}\,\partial_{a\beta} + \hat{M}^{b}_{a}\theta_{b\alpha} - 2\hat{N}^{\beta}_{\alpha}\theta_{a\beta}\\
M_{ab} &= \theta^{\alpha}_{a}\,\partial_{b\alpha} + \theta^{\alpha}_{b}\,\partial_{a\alpha} - \hat{M}_{ab}\\
N_{\alpha\beta} &= \theta^{a}_{\alpha}\,\partial_{a\beta} + \theta^{a}_{\beta}\,\partial_{a\alpha} - \hat{N}_{\alpha\beta}.
\end{aligned} \tag{22}$$

As in the examples discussed in § 2, examination of the $L_\pm$ and $Q_{\dot{2}a\alpha}$ action shows that the finite-dimensional factor (polynomial in both x and θ) is most readily seen in terms of a modified basis of component fields whose definitions and degrees (highest power of x in the finite factor) are A, $2L$; ψ^{mn}, $2L-1$; F^m, $\tilde{F}^0$, G^n, $\tilde{G}^0$, $2L-2$; $\tilde{\chi}^{mn}$, $2L-3$; $\tilde{D}$, $2L-4$; where

$$\begin{aligned}
\tilde{F}^0_{ab} &= F^0_{ab} - \hat{M}_{ab}A'/4L \\
\tilde{G}^0_{\alpha\beta} &= G^0_{\alpha\beta} + \hat{N}_{\alpha\beta}A'/2L \\
\tilde{\chi}^{mn} &= \chi^{mn} + (3+2M^m+4N^n)\psi^{mn'}/6(L-\tfrac{1}{2}) \\
\tilde{D} &= D + \hat{M}^{ab}F^{0\prime}_{ab}/12(L-1) + \hat{N}^{\alpha\beta}G^{0\prime}_{\alpha\beta}/6(L-1) \\
&\qquad -[3L+2M(M+1)-8N(N+1)-3]A''/12L(L-\tfrac{1}{2})(L-1)
\end{aligned} \tag{23}$$

with $m, n = \bullet$ and $2M^\pm + 1 = \pm(2M+1)$, etc (see appendix 1 for details of the projection operators).

From (22), taking into account the definitions (23), the $O(4)\times Sp(2) = SU(2)\times SU(2)\times SU(2)$ decompositions we obtain for arbitrary induced rpresentations with the chosen little group (corresponding to superfields of arbitrary half-integer charge L, and spins M and N) are given in table 3. This class of irreducible representations is in general typical (with even and odd dimensions the same), and total dimension $16(2L-1)(2M+1)(2N+1)$, with $L \geq \frac{3}{2}$, M, $N \geq 0$. In the basis (23) it is found that superfields which cannot be decomposed arise for certain (L, M, N) values, corresponding to atypical representations (cf Kac 1978). For example, in the case $(L = 2N+1, M = 0, N \geq \frac{1}{2})$, the set $\tilde{G}^-$, $\tilde{D}$ and $\tilde{\chi}^{+-}$ form an invariant subspace of dimension $32N^2 - 2 = 16N^2 - 2/16N^2$; on the other hand, with $(L = 2N+2, M = 0, N \geq 0)$ the analogous set A, G^+ and ψ^{++} are invariant as a factor space. From (23) it is also evident that $L = 0, \frac{1}{2}, 1$ are special cases. For example, the sequence $(L = 1, M = 0, N \geq 0)$ with invariant set A, F^+, $G^\pm$ and $\psi^{+\pm}$, includes the adjoint $\mathbf{17} = (\mathbf{3}\times\mathbf{1}\times\mathbf{1}+\mathbf{1}\times\mathbf{3}\times\mathbf{1}+\mathbf{1}\times\mathbf{1}\times\mathbf{3})/(\mathbf{2}\times\mathbf{2}\times\mathbf{2})$ for $N = 0$. From Kac (1978), the general atypicality conditions are $L = M^\pm - 2N^\pm$, $L = M^\pm - 2N^\mp$, but we have not worked out the general decompositions. Likewise the connection with Young diagrams has not been made explicit, apart from special cases, for example that the $(L = 1, M = 0, N \geq 0)$ sequence corresponds to rank-$(2N+2)$ graded antisymmetrical tensors (see, for example, Dondi and Jarvis 1981).

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Appendix 1. Projection operators for spin $\mathbf{M}\times\frac{1}{2}$ and spin $\mathbf{M}\times\mathbf{1}$

As explained in § 2, the two-index basis for SU(2) used there and in §§ 3 and 4 is related to the spherical basis via

$$\hat{M}_{\alpha\beta} = 2(\hat{\boldsymbol{M}}\cdot\boldsymbol{\sigma}\varepsilon)_{\alpha\beta}$$

where the matrices refer to a spin-M representation of an SU(2) little group. Where these act on superfield components such as ψ_α or $P_{(\alpha\beta)}$, the question arises of projections onto total spins $M\pm\frac{1}{2}$, or $M\pm 1$, $M+0$ respectively. These are derived straightforwardly using the characteristic identity (quadratic or cubic, respectively) satisfied by the generators in the reducible $M\times\frac{1}{2}$, $M\times 1$ representations.

For $M\times\frac{1}{2}$ we have (spin-M indices are suppressed and indices $\alpha, \beta, \ldots$ are raised using the inverse metric $\varepsilon^{\alpha\beta}$)

$$\begin{aligned} \Pi_{\alpha}^{\pm\frac{1}{2}\,\beta} &= (\hat{M}_{\alpha}^{\beta} - 2M^{\mp}\,\delta_{\alpha}^{\beta})/2(2M^{\pm}+1) \\ \delta_{\alpha}^{\beta} &= \Pi_{\alpha}^{+\frac{1}{2}\,\beta} + \Pi_{\alpha}^{-\frac{1}{2}\,\beta} \\ \hat{M}_{\alpha}^{\beta} &= 2M^{+}\Pi_{\alpha}^{+\frac{1}{2}\,\beta} + 2M^{-}\Pi_{\alpha}^{-\frac{1}{2}\,\beta} \end{aligned} \tag{A1}$$

where $M^{+}=M$ and $M^{-}=-M-1$.

For $M\times 1$ we introduce

$$\begin{aligned} 1_{\alpha\beta}^{\gamma\delta} &= \tfrac{1}{2}(\delta_{\alpha}^{\gamma}\,\delta_{\beta}^{\delta} + \delta_{\alpha}^{\delta}\,\delta_{\beta}^{\gamma}) \\ \hat{M}_{\alpha\beta}^{\gamma\delta} &= \tfrac{1}{4}(\hat{M}_{\alpha}^{\gamma}\,\delta_{\beta}^{\delta} + \delta_{\alpha}^{\gamma}\hat{M}_{\beta}^{\delta} + \hat{M}_{\beta}^{\gamma}\,\delta_{\alpha}^{\delta} + \delta_{\beta}^{\gamma}\hat{M}_{\alpha}^{\delta}) \\ \hat{N}_{\alpha\beta}^{\gamma\delta} &= \tfrac{1}{4}(\hat{M}_{\alpha}^{\gamma}\hat{M}_{\beta}^{\delta} + \hat{M}_{\alpha}^{\delta}\hat{M}_{\beta}^{\gamma} + \hat{M}_{\beta}^{\gamma}\hat{M}_{\alpha}^{\delta} + \hat{M}_{\beta}^{\delta}\hat{M}_{\alpha}^{\gamma}) \end{aligned}$$

and the projectors are

$$\begin{aligned} \Pi_{\alpha\beta}^{\pm 1\,\gamma\delta} &= \left(\frac{\hat{N} + (2M^{\pm}+3)\hat{M} + (M^{\pm}+1)(M^{\pm}+2)}{2(M^{\pm}+1)(2M^{\pm}+1)}\right)_{\alpha\beta}^{\gamma\delta} \\ \Pi_{\alpha\beta}^{0\,\gamma\delta} &= \left(\frac{\hat{N}+\hat{M}+M^{+}M^{-}}{M^{+}M^{-}}\right)_{\alpha\beta}^{\gamma\delta}. \end{aligned} \tag{A2}$$

From these definitions, several useful identities can be derived which are necessary for the extraction of component field variations. Examples are

$$\begin{aligned} \Pi_{\alpha}^{\pm\frac{1}{2}\,\gamma}\,\delta_{\beta}^{\delta}P_{\gamma\delta}^{\pm 1} &= P_{\alpha\beta}^{\pm 1} \\ \Pi_{\alpha}^{\pm\frac{1}{2}\,\gamma}\hat{M}_{\beta}^{\delta}P_{\gamma\delta}^{\pm 1} &= 2M^{\pm}P_{\alpha\beta}^{\pm 1} \\ \Pi_{\alpha}^{\pm\frac{1}{2}\,\gamma}\,\delta_{\beta}^{\delta}P_{\gamma\delta}^{\mp 1} &= 0 = \Pi_{\alpha}^{\pm\frac{1}{2}\,\gamma}\hat{M}_{\beta}^{\delta}P_{\gamma\delta}^{\mp 1} \\ \Pi_{\alpha}^{\pm\frac{1}{2}\,\gamma}\,\delta_{\beta}^{\delta}P_{\gamma\delta}^{0} &= [M^{\pm}P_{\alpha\beta}^{0} + \tfrac{1}{4}\varepsilon_{\alpha\beta}\hat{P}^{0}]/(2M^{\pm}+1) \\ \Pi_{\alpha}^{\pm\frac{1}{2}\,\gamma}\hat{M}_{\beta}^{\delta}P_{\gamma\delta}^{0} &= -2[M^{\pm}P_{\alpha\beta}^{0} + \tfrac{1}{4}\varepsilon_{\alpha\beta}\hat{P}^{0}](M^{\pm}+2)/(2M^{\pm}+1) \end{aligned} \tag{A3}$$

where $P^{\pm 1}$ and P^{0} satisfy

$$\Pi_{\alpha\beta}^{\pm 1\,\gamma\delta}P_{(\gamma\delta)}^{\pm 1} = P_{(\alpha\beta)}^{\pm 1} \qquad \Pi_{\alpha\beta}^{0\,\gamma\delta}P_{(\gamma\delta)}^{0} = P_{(\alpha\beta)}^{0}$$

and

$$\hat{P}^{0} \equiv \hat{M}^{\alpha\beta}P_{\alpha\beta}^{0}.$$

Other examples are

$$\begin{aligned} \tfrac{1}{2}(\delta_{\alpha}^{\gamma}\,\delta_{\beta}^{\delta} + \delta_{\beta}^{\gamma}\,\delta_{\alpha}^{\delta})\eta_{\gamma}\psi_{\delta}^{\pm\frac{1}{2}} &= (\Pi_{\alpha\beta}^{\pm 1\,\gamma\delta} + \Pi_{\alpha\beta}^{0\,\gamma\delta})\eta_{\gamma}\psi_{\delta}^{\pm\frac{1}{2}} \\ \tfrac{1}{2}(\hat{M}_{\alpha}^{\gamma}\,\delta_{\beta}^{\delta} + \hat{M}_{\beta}^{\gamma}\,\delta_{\alpha}^{\delta})\eta_{\gamma}\psi_{\delta}^{\pm\frac{1}{2}} &= 2M^{\pm}\Pi_{\alpha\beta}^{\pm 1\,\gamma\delta} - 2(M^{\pm}+2)\Pi_{\alpha\beta}^{0\,\gamma\delta}\eta_{\gamma}\psi_{\delta}^{\pm\frac{1}{2}} \end{aligned} \tag{A4}$$

where $\Pi_{\alpha}^{\pm\frac{1}{2}\,\beta}\psi_{\beta}^{\pm\frac{1}{2}} = \psi_{\alpha}^{\pm\frac{1}{2}}$ and η_{β} is a spinor parameter.

Finally we have

$$\begin{aligned} &\Pi_{\alpha\beta}^{\pm 1\,\gamma\delta}\hat{M}_{\gamma\delta} = 0 = \hat{M}^{\alpha\beta}\Pi_{\alpha\beta}^{\pm 1\,\gamma\delta} \\ &\Pi_{\alpha\beta}^{0\,\gamma\delta}\hat{M}_{\gamma\delta} = \hat{M}_{\alpha\beta} \qquad \hat{M}^{\alpha\beta}\Pi_{\alpha\beta}^{0\,\gamma\delta} = \hat{M}^{\gamma\delta}. \end{aligned} \tag{A5}$$

Appendix 2: Matrix representations of OSp(1/2)

In §§ 2.2 and 3 the little groups chosen involve the supergroup OSp(1/2) with generators $\{M_{\alpha\beta}, Q_\alpha\}$ as in § 2.1. The superfield technique requires explicit matrix elements $\{\hat{\mathcal{M}}_{\alpha\beta}, \hat{\mathcal{Q}}_\alpha\}$ of these generators acting on superfields of arbitrary 'superspin' M (§ 2.1), i.e. two-component superfields

$$\Phi_A = \begin{pmatrix} \phi_a \\ \phi_{a\alpha} \end{pmatrix} \tag{A6}$$

where ϕ_a has spin M and $\phi_{a\alpha}$ has spin $M-\frac{1}{2}$, or

$$(\Pi^{-\frac{1}{2}})_{a\alpha}^{b\beta}\phi_{b\beta} = \phi_{a\alpha}.$$

For the Sp(2) spin generators we take the matrices

$$(\hat{\mathcal{M}}_{\alpha\beta})_C^D = \begin{pmatrix} (\hat{M}_{\alpha\beta})_c^d & 0 \\ 0 & (\hat{M}^x_{\alpha\beta})_{c\gamma}^{d\delta} \end{pmatrix} \tag{A7}$$

where $(\hat{M}_{\alpha\beta})_c^d$ are the matrix reprexentatives for spin M, and $(\hat{M}^x_{\alpha\beta})_{c\gamma}^{d\delta}$ correspond to the reducible $M\times\frac{1}{2}$ representation:

$$(\hat{M}^x_{\alpha\beta})_{c\gamma}^{d\delta} = (\hat{M}_{\alpha\beta})_c^d\,\delta_\gamma^\delta + \varepsilon_{\gamma\alpha}\,\delta_\beta^\delta + \varepsilon_{\gamma\beta}\,\delta_\alpha^\delta.$$

That this choice is appropriate is guaranteed by the fact that the spin $M\pm\frac{1}{2}$ projectors commute with the $\hat{M}^x_{\alpha\beta}$, so

$$\begin{aligned}\Pi_{\gamma c}^{-\frac{1}{2}\ \delta d}(\hat{M}^x_{\alpha\beta})_{\delta d}^{\varepsilon e}\phi_{e\varepsilon} &= (\hat{M}^x_{\alpha\beta})_{\gamma c}^{d\delta}(\Pi^{-\frac{1}{2}})_{d\delta}^{e\varepsilon}\phi_{e\varepsilon} \\ &\equiv (\hat{M}^x_{\alpha\beta})_{\gamma c}^{d\delta}\phi_{d\delta}\end{aligned}$$

because $\phi_{a\alpha}$ has spin $M-\frac{1}{2}$.

From the anticommutation relations we require matrices $(\hat{\mathcal{Q}}_\alpha)_C^D$ which satisfy

$$(\hat{\mathcal{Q}}_\alpha)_C^D(\hat{\mathcal{Q}}_\beta)_D^E + (\hat{\mathcal{Q}}_\beta)_C^D(\hat{\mathcal{Q}}_\alpha)_D^E = -(\hat{\mathcal{M}}_{\alpha\beta})_C^E$$

and we find (here $[M]\equiv(2M+1)^{1/2}$)

$$(\hat{\mathcal{Q}}_\alpha)_B^C = [M]\begin{pmatrix} 0 & \Pi_{\alpha b}^{-\frac{1}{2}\ \gamma c} \\ (\Pi^{-\frac{1}{2}}\varepsilon)_{b\beta\alpha}^{c} & 0 \end{pmatrix}. \tag{A8}$$

When working with $\hat{\mathcal{Q}}_\alpha$ we must be careful to ensure that it anticommutes with a-numbers, though from the form (A8) this is not explicit. Finally the action of $(\hat{\mathcal{M}}_{\alpha\beta})_C^D$ and $(\hat{\mathcal{Q}}_\alpha)_C^D$ on the superfield Φ_D is

$$(\hat{\mathcal{M}}_{\alpha\beta})_C^D\Phi_D = \begin{pmatrix} (\hat{M}_{\alpha\beta})_c^d\phi_d \\ (\hat{M}^x_{\alpha\beta})_{c\gamma}^{d\delta}\phi_{d\delta} \end{pmatrix}$$

and

$$(\hat{\mathcal{Q}}_\alpha)_C^D\Phi_D = \begin{pmatrix} \phi_{\gamma c} \\ (\Pi^{-\frac{1}{2}}\varepsilon)_{\gamma\alpha}\phi_d \end{pmatrix}. \tag{A9}$$

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